ELSEVIER

Journal of Geometry and Physics 45 (2003) 309–322

JOURNAL OF GEOMETRY AND PHYSICS

www.elsevier.com/locate/jgp

A local analytic characterization of Schwarzschild metrics

Manuel Fernández-López [a], Eduardo García-Río [a,*], Demir N. Kupeli [b]

[a] *Department of Geometry and Topology, Faculty of Mathematics, University of Santiago de Compostela, E-15782 Santiago de Compostela, Spain*

[b] *Department of Mathematics, Atılım University, İncek, 06836 Ankara, Turkey*

Received 15 May 2001

Abstract

A local characterization of Schwarzschild metrics is made by showing the space-time is locally a 2 by 2 warped product and admitting a static reference frame on its certain open subsets under some assumptions on the global analytic structure and stress-energy tensor of the space-time, such as, assuming the existence of solutions to certain partial differential equations and the existence of a radiation stress-energy tensor consistent with these solutions on the space-time.

MSC: 53C50; 58Z05; 83C75

Subj. Class.: General relativity

Keywords: Local Möbius equation; Warped product decomposition; Tension field of a vector field; Static reference frame

1. Introduction

Physical observations are made locally and hence they are used as a basis to confirm or raise a space-time model. In the presence of no matter, physical observations are mostly made on certain components of the curvature tensor of the space-time and suited to build a model. In [6], by putting conditions on certain components of the curvature tensor of a space-time (which are consistent with physical observations), a local characterization of Schwarzschild and Reissner solutions are made. In this paper, we follow a completely

* Corresponding author.
E-mail addresses: manufl@usc.es (M. Fernández-López), xtedugr@usc.es (E. García-Río), dnkupeli@superonline.com (D.N. Kupeli).

PII: S0393-0440(01)00063-8

different approach to local characterization of a "static star" by putting conditions on the global analytic structure of space-time together with some mild assumptions on its Ricci tensor, in fact on the stress-energy tensor of space-time. That is, we remove the assumptions on the curvature tensor, especially the ones in [6], and compensate this lack by the analytic assumptions together with the assumptions on the stress-energy tensor. (A reminiscent approach to characterize the cosmological time functions is made in [3].)

Intuitively, a "static star" refers to a gravitational field generated by a time independent non-rotating source. However, throughout this paper, we only consider the gravitational field exterior to the celestial body of a "static star" with no matter is present, yet an electromagnetic radiation may be present, and still call that gravitation a "static star".

In Section 2, we introduce a partial differential equation, called the local Möbius equation. In fact, the existence of a submersive solution to this equation together with the existence of a "consistent" radiation stress-energy tensor to this solution, will yield the major local 2 by 2 warped product decomposition of a Schwarzschild space-time in Section 3. Yet, the existence of a static reference frame is considered to be an essential ingredient of a "static star" model. We also show the existence of such a reference frame on some open subsets of space-time by imposing some further conditions on the solution of the local Möbius equation implicitly, which is, in fact, another partial differential equation on the space-time as well. However, explicitly it is the condition called weak-affinity in [6]. Throughout this paper, everything at hand is assumed to be smooth.

2. The local Möbius equation

The Möbius equation was first defined on Riemannian manifolds by Osgood and Stowe [12]. This equation can also be defined on semi-Riemannian manifolds as follows: let (M, g) be an n-dimensional semi-Riemannian manifold and $f:(M, g) \to \mathbb{R}$ be a function. Then f is said to satisfy the Möbius equation on (M, g) if

$$H_f + \mathrm{d}f \otimes \mathrm{d}f - \frac{1}{n}(\Delta f + g(\nabla f, \nabla f))g = 0$$

or equivalently, $h_f + g(\cdot, \nabla f)\nabla f - (1/n)(\Delta f + g(\nabla f, \nabla f))id = 0$, where H_f and h_f are the Hessian form and tensor of f on (M, g), respectively, and ∇f and Δf $(= \mathrm{div}\nabla f)$ are the gradient and Laplacian of f on (M, g), respectively. It is easy to show that, if $f:(M, g) \to \mathbb{R}$ satisfies the Möbius equation on (M, g) then the function $t = \mathrm{e}^f$ satisfies the equation

$$H_t = \frac{\Delta t}{n} g$$

or equivalently

$$h_t = \frac{\Delta t}{n} id$$

on (M, g). Although this linearization of the Möbius equation is not equivalent to the Möbius equation globally, they are locally equivalent considering the existence of solutions to these equations. Hence the equation defined by $H_t = (\Delta t/n)g$ for functions $t:(M, g) \to \mathbb{R}$ may be called the local Möbius equation. This equation is extensively studied in Riemannian and semi-Riemannian geometry in connection with conformal diffeomorphisms and local warped product decompositions (see [8] and the references therein).

From viewpoint of general relativity, either of the Möbius or local Möbius equation does not seem to be of interest since physically meaningful functions, such as time functions, on physically realistic space-times do not satisfy either of these equations. That is why, in [3], the Möbius equation is modified to be satisfied by certain time functions on physically realistic space-times to characterize the cosmological time functions by being its solutions. That is, a time function $f:(M, g) \to \mathbb{R}$ on an n-dimensional space-time is said to satisfy the Möbius equation if

$$H_f(X, Y) - \frac{1}{n-1}(\Delta f + g(\nabla f, \nabla f))g(X, Y) = 0 \quad \text{and} \quad H_f(Z, X) = 0$$

or equivalently, $[h_f - (1/(n-1))(\Delta f + g(\nabla f, \nabla f))id]X = 0$ for every $X, Y \in \Gamma \ker f_*$, $Z \in \Gamma(\ker f_*)^\perp$. Then again, if a time function $f:(M, g) \to \mathbb{R}$ satisfies the above Möbius equation then the function $t = \mathrm{e}^f$ satisfies the local Möbius equation

$$H_t(X, Y) = \frac{\Delta t}{n-1} g(X, Y) \quad \text{and} \quad H_t(Z, X) = 0$$

or equivalently, $h_t(X) = (\Delta t/(n-1))X$, for every $X, Y \in \Gamma \ker f_*$, $Z \in \Gamma(\ker f_*)^\perp$. It is also shown in [3] that the time functions on a space-time (M, g) characterized by being solutions of the Möbius equation yield local Lorentzian warped product decompositions of (M, g) as in cosmological space-times, provided that the stress-energy tensor of (M, g) is a fluid "consistent" with f.

Naturally, one may ask whether the Möbius or local Möbius equation can be used to characterize the functions on a 4-dimensional space-time which yield a local 2 by 2 warped product decomposition of (M, g) as in Schwarzschild models. Indeed this may be done by generalizing the local Möbius equation.

Let (M_1, g_1) and (M_2, g_2) be semi-Riemannian manifolds with dimensions $\dim M_1 = n_1 > n_2 = \dim M_2 \geq 1$, and let $f:(M_1, g_1) \to (M_2, g_2)$ be a submersion with semi-Riemannian fibers in (M_1, g_1), that is, the fibers are semi-Riemannian submanifolds of (M_1, g_1) in the induced structure from g_1. Then f is said to satisfy the local Möbius equation if

$$(\nabla f_*)(X, Y) = \frac{\tau(f)}{n_1 - n_2} g(X, Y) \quad \text{and} \quad (\nabla f_*)(W, X) = 0$$

for every $X, Y \in \Gamma \ker f_*$, $W \in \Gamma(\ker f_*)^\perp$, where ∇f_* and $\tau(f)$ are the second fundamental form and tension field of f, respectively (see for example [5,9] for more about ∇f_* and $\tau(f)$). In this paper, we deal with a special case of the above definition, that is, we take (M_1, g_1) and (M_2, g_2) to be 4-dimensional space-time (M, g) and the Euclidean space $(\mathbb{R}^2, \sum_{i=1}^2 \mathrm{d}x^i \otimes \mathrm{d}x^i)$, respectively, where (x^1, x^2) are the usual coordinates on $\mathbb{R}^2$.

First note that, if (M, g) is a semi-Riemannian manifold and $f = (f_1, \dots, f_m) : (M, g) \to (\mathbb{R}^m, \sum_{i=1}^m \mathrm{d}x^i \otimes \mathrm{d}x^i)$ is a map, then

$$(\nabla f_*)(X, Y) = \sum_{i=1}^m H_{f_i}(X, Y) \frac{\partial}{\partial x^i} \circ f$$

for every $X, Y \in \Gamma TM$, and hence

$$\tau(f) = \sum_{i=1}^{m} (\Delta f_i) \frac{\partial}{\partial x^i} \circ f$$

Proposition 1. *Let* (M, g) *be a* 4*-dimensional space-time and* $f = (f_1, f_2) : (M, g) \to (\mathbb{R}^2, \sum_{i=1}^{2} \mathrm{d}x^i \otimes \mathrm{d}x^i)$ *be a submersion with Riemannian fibers in* (M, g)*. If f satisfies the local Möbius equation on* (M, g) *then*

1. $H_{f_i}(X, Y) = (\Delta f_i/2) g(X, Y)$ *and* $H_{f_i}(W, X) = 0$, *or equivalently*

   $$h_{f_i}(X) = \tfrac{1}{2}(\Delta f_i) X$$

   for every $X, Y \in \Gamma \ker f_*$, $W \in \Gamma(\ker f_*)^{\perp}$.
2. $H_{f_i}(Z, Z) = H_{f_i}(W, W)$ *for every orthonormal* $Z, W \in \Gamma(\ker f_*)^{\perp}$.

Proof.

1. Obvious.
2. Let $Z, W \in \Gamma(\ker f_*)^{\perp}$ and $X, Y \in \Gamma(\ker f_*)^{\perp}$ such that $\{Z, W, X, Y\}$ is a local orthonormal frame in *TM*. Then

   $$\begin{aligned} \tau(f) &= g(Z, Z)(\nabla f_*)(Z, Z) + g(W, W)(\nabla f_*)(W, W) \\ &\quad + (\nabla f_*)(X, X) + (\nabla f_*)(Y, Y) \\ &= g(Z, Z)(\nabla f_*)(Z, Z) + g(W, W)(\nabla f_*)(W, W) + \tau(f) \end{aligned}$$

   Thus $g(Z, Z)(\nabla f_*)(Z, Z) + g(W, W)(\nabla f_*)(W, W) = 0$ and it follows from (1) and $g(Z, Z) = -g(W, W)$ that $H_{f_i}(Z, Z) = H_{f_i}(W, W)$. □

Remark 1. Note in the above proposition that (1) is equivalent to *f* is a solution of the local Möbius equation.

3. A characterization of Schwarzschild metrics

3.1. Decomposition theorems

A 4-dimensional space-time (M, g) is said to obey the Einstein equation for a stress-energy tensor T if $\mathrm{Ric} - (1/2)(\mathrm{Sc})g = T$, where Ric and (Sc) are the Ricci tensor and scalar curvature of (M, g), respectively. A stress-energy tensor T on (M, g) is called a 'radiation' if $trT = 0$, that is, no matter is present. Note that, if (M, g) obeys the Einstein equation for a radiation stress-energy tensor T then $\mathrm{Ric} = T$.

Now let (M, g) be a 4-dimensional space-time and $f:(M, g) \to \mathbb{R}^2$ be a submersion with Riemannian fibers in (M, g). Suppose (M, g) obeys the Einstein equation for a radiation stress-energy tensor T given by

$$T = \rho(-g|_{(\ker f_*)^{\perp}} \oplus g|_{(\ker f_*)})$$

where $g|_{(\ker f_*)^\perp}$ and $g|_{(\ker f_*)}$ are the restrictions of g to $(\ker f_*)^\perp$ and $(\ker f_*)$, respectively, and ρ is a function on M, referred as the negatives of the Faraday stresses ([4], p. 124). Note that, if $\rho = 0$, then (M, g) is called vacuum.

Now we are ready to characterize the Schwarzschild metrics by the existence of certain functions on spacetimes. Note also in the statements of the theorems below that, by a "static star", we mean the gravitational field exterior to the celestial body of a "static star" since T is taken to be a radiation stress-energy tensor, as noted before.

Theorem 1. *Let (M, g) be a 4-dimensional space-time and $f = (f_1, f_2) : (M, g) \to (\mathbb{R}^2, \sum_{i=1}^2 \mathrm{d}x^i \otimes \mathrm{d}x^i)$ be a submersion with Riemannian fibers in (M, g). Suppose (M, g) satisfies the Einstein equation for a stress-energy tensor*

$$T = \rho(-g|_{(\ker f_*)^\perp} \oplus g|_{(\ker f_*)})$$

and f satisfies the local Möbius equation on (M, g).

Then (M, g) is locally a warped product $(M_1 \times M_2, g_1 \oplus \psi^2 g_2)$, where (M_1, g_1) is a Lorentzian surface and (M_2, g_2) is a Riemannian surface of constant curvature.

Proof. First we show that $(\ker f_*)^\perp$ is a totally geodesic distribution. For this, it suffices to show that $\nabla_{\nabla f_j} \nabla f_i \in \Gamma(\ker f_*)^\perp$ for $i, j = 1, 2$. Now let $X \in \Gamma \ker f_*$. Then

$$g(\nabla_{\nabla f_j} \nabla f_i, X) = g(\nabla_X \nabla f_i, \nabla f_j) = \tfrac{1}{2}(\Delta f_i) g(X, \nabla f_j) = 0$$

and hence $\nabla_{\nabla f_j} \nabla f_i \in \Gamma(\ker f_*)^\perp$ in showing that $(\ker f_*)^\perp$ is a totally geodesic distribution with totally geodesic integral manifolds.

Next we show that the fibers of f are totally umbilical and spherical. Let $\mathbb{I}$ be the second fundamental form tensor of the fibers of f. Then, since ∇f_1 and ∇f_2 are linearly independent at each $p \in M$ and $g(\mathbb{I}(X, Y), \nabla f_i) = -(\Delta f_i/2) g(X, Y)$ for every $X, Y \in \Gamma \ker f_*$, it can be shown with a straightforward computation that

$$\begin{aligned}\mathbb{I}(X, Y) = \frac{1}{2Q(\nabla f_1, \nabla f_2)}&[(g(\nabla f_1, \nabla f_2)\Delta f_2 - g(\nabla f_2, \nabla f_2)\Delta f_1)\nabla f_1 \\ &+ (g(\nabla f_1, \nabla f_2)\Delta f_1 - g(\nabla f_1, \nabla f_1)\Delta f_2)\nabla f_2] g(X, Y)\end{aligned}$$

where $Q(\nabla f_1, \nabla f_2) = g(\nabla f_1, \nabla f_1) g(\nabla f_2, \nabla f_2) - g(\nabla f_1, \nabla f_2)^2$. Thus the fibers of f are totally umbilical and their mean curvature vector field N is given by

$$\begin{aligned}N = \frac{1}{Q(\nabla f_1, \nabla f_2)}&[(g(\nabla f_1, \nabla f_2)\Delta f_2 - g(\nabla f_2, \nabla f_2)\Delta f_1)\nabla f_1 \\ &+ (g(\nabla f_1, \nabla f_2)\Delta f_1 - g(\nabla f_1, \nabla f_1)\Delta f_2)\nabla f_2].\end{aligned}$$

Now, to show that the fibers of f are spherical, we have to show that the mean curvature vector field N is normal parallel. For this, observe from the expression of N in terms of the functions f_1 and f_2 that, it suffices to show Δf_1 and Δf_2 are constant on each fiber of f.

Now let $Z, W \in \Gamma(\ker f_*)^\perp$ and $X, Y \in \Gamma \ker f_*$ be such that $\{Z, W, X, Y\}$ is a local orthonormal frame in TM. Then, since $\mathrm{Ric} = T$

$$\begin{aligned}0 &= \mathrm{Ric}(Y, \nabla f_i) = g(Z, Z) g(R(Z, Y)\nabla f_i, Z) + g(W, W) g(R(W, Y)\nabla f_i, W) \\ &\quad + g(R(X, Y)\nabla f_i, X).\end{aligned}$$

Now

$$\begin{aligned} R(X,Y)\nabla f_i &= \nabla_X\nabla_Y\nabla f_i - \nabla_Y\nabla_X\nabla f_i - \nabla_{[X,Y]}\nabla f_i \\ &= \tfrac{1}{2}\{\nabla_X((\Delta f_i)Y) - \nabla_Y((\Delta f_i)X) - (\Delta f_i)[X,Y]\} \\ &= \tfrac{1}{2}\{X(\Delta f_i)Y - Y(\Delta f_i)X\} \end{aligned}$$

and hence $g(R(X,Y)\nabla f_i, X) = -(1/2)Y(\Delta f_i)$.

Also, since $(\ker f_*)^{\perp}$ is a totally geodesic distribution orthogonal to $\ker f_*$

$$g(R(Z,Y)\nabla f_i, Z) = g(R(\nabla f_i, Z)Z, Y) = 0$$

and

$$g(R(W,Y)\nabla f_i, W) = g(R(\nabla f_i, W)W, Y) = 0.$$

Thus $0 = \mathrm{Ric}(Y, \nabla f_i) = -(1/2)Y(\Delta f_i)$, and it follows that Δf_i is constant on each fiber of f. That is, the fibers of f are spherical as well. Hence it follows from [14] (proposition 3-d) that (M, g) is locally a warped product $(M_1 \times M_2, g_1 \oplus \psi^2 g_2)$, where (M_1, g_1) is a Lorentzian surface and (M_2, g_2) is a Riemannian surface. Now we show that each fiber of f is of constant curvature. For this, we compute the Ricci tensor of a fiber M_2 of f in its induced Riemannian structure from (M, g). From [11] (p. 211) (see also [1]), for $X, Y \in \Gamma TM_2$

$$\mathrm{Ric}_{M_2}(X,Y) = \mathrm{Ric}(X,Y) + \left(\frac{\Delta\psi}{\psi} + \frac{g(\nabla\psi,\nabla\psi)}{\psi^2}\right) g(X,Y)$$

where Ric_{M_2} is the Ricci tensor of the induced Riemannian structure of a fiber M_2 of f from (M, g) and, $\nabla\psi$ and $\Delta\psi$ are the gradient and Laplacian of the warping function ψ in the induced Lorentzian structure of M_1 from (M, g). Thus, by $\mathrm{Ric} = T$, we obtain

$$\mathrm{Ric}_{M_2}(X,Y) = \left(\rho + \frac{\Delta\psi}{\psi} + \frac{g(\nabla\psi,\nabla\psi)}{\psi^2}\right) g(X,Y) = \frac{\Lambda}{2} g(X,Y)$$

and Λ is constant on each fiber M_2 of f since ρ and $(\Delta\psi/\psi) + (g(\nabla\psi,\nabla\psi)/\psi^2)$ are constant on each fiber of f (note here that, since $(\ker f_*)^{\perp}$ is a totally geodesic distribution, it follows from $0 = (\mathrm{div}T)(X) = \mathrm{d}\rho(X)$ for every $X \in \Gamma\ker f_*$ that ρ is constant on each fiber of f). Thus, each fiber of f is of constant curvature in its induced Riemannian structure from (M, g). □

Note here that, if the fibers of f are compact, M can be locally written as a product $M_1 \times M_2$, where M_2 is a fiber of f and M_1 is an open subset of an integral manifold of $(\ker f_*)^{\perp}$.

Remark 2. Note that the expression $(\Delta\psi/\psi) + (g(\nabla\psi,\nabla\psi)/\psi^2)$ is equal to $-\mathrm{div}N$ in the above proof. Indeed from [11] (p. 206), since $N = -\nabla\psi/\psi$, we can compute this expression in terms of N. First note that, $(g(\nabla\psi,\nabla\psi)/\psi^2) = g(N,N)$ and

$$\frac{\Delta\psi}{\psi} = -\frac{1}{\psi}\mathrm{div}_{M_1}(\psi N) = -\frac{1}{\psi}[g(\nabla\psi, N) + \psi\,\mathrm{div}_{M_1} N] = g(N,N) - \mathrm{div}_{M_1} N$$

along M_1, where $\mathrm{div}_{M_1} N$ is the divergence of N in the induced Lorentzian structure of M_1 from (M, g). Also, since the fibers of f are totally umbilical

$$\mathrm{div}N = \mathrm{div}_{M_1} N + g(\nabla_X N, X) + g(\nabla_Y N, Y) = \mathrm{div}_{M_1} N - 2g(N, N)$$

where $X, Y \in \Gamma \ker f_*$ are orthonormal vector fields along M_2, and it follows that $(\Delta\psi/\psi) = -g(N, N) - \mathrm{div}N$. That is, $(\Delta\psi/\psi) + (g(\nabla\psi, \nabla\psi)/\psi^2) = -\mathrm{div}N$. Consequently, we also obtain $(\mathrm{Sc})_{M_2} = 2(\rho - \mathrm{div}\, N)$, where $(\mathrm{Sc})_{M_2}$ is the scalar curvature of M_2 in its induced Riemannian structure from (M, g).

Remark 3. Next we state a theorem to determine the geometry of a Schwarzschild space-time on the basis of its global analytic structure. Note that in the statement of Theorem 2, instead of assuming the existence of the function φ, we can assume that $\Lambda > 0$ on M. Then, since all compact, orientable, constant positive curvature Riemannian surfaces are homothetic to the standard Euclidean 2-sphere, the conclusion of Theorem 2 remains valid. Again, instead of assuming the existence of the function φ, we can assume the fibers of f are simply connected. Since all simply connected, compact, constant curvature Riemannian surfaces are homothetic to the standard Euclidean 2-sphere, the conclusion of Theorem 2 remains valid. The reason we assumed the existence of the function φ in the statement of Theorem 2, is to determine the geometry by imposing conditions on the global analytic structure of space-time, rather than its geometric or topological structures as in the above alternative assumptions, respectively.

Theorem 2. *Let (M, g) be a 4-dimensional space-time and $f = (f_1, f_2) : (M, g) \to (\mathbb{R}^2, \sum_{i=1}^{2} \mathrm{d}x^i \otimes \mathrm{d}x^i)$ be a submersion with compact Riemannian fibers in (M, g). Suppose (M, g) satisfies the Einstein equation for a stress-energy tensor $T = \rho(-g|_{(\ker f_*)^\perp} \oplus g|_{(\ker f_*)})$, and f satisfies the local Möbius equation on (M, g). If there exists a function $\varphi: M \to \mathbb{R}$ which is not constant on each fiber of f and satisfies the equation*

$$\begin{aligned}\Delta\varphi &+ 2g(\nabla\varphi, N) + \frac{1}{Q(\nabla f_1, \nabla f_2)}[(-g(\nabla f_1, \nabla f_1)^2 g(\nabla f_2, \nabla f_2)^2 \\ &+ 2g(\nabla f_1, \nabla f_1) g(\nabla f_1, \nabla f_2)^2 - g(\nabla f_1, \nabla f_1)^3) H_\varphi(\nabla f_1, \nabla f_1) \\ &+ g(\nabla f_1, \nabla f_2) Q(\nabla f_1, \nabla f_2) H_\varphi(\nabla f_1, \nabla f_2) + (-g(\nabla f_2, \nabla f_2)^2 g(\nabla f_1, \nabla f_1)^2 \\ &+ 2g(\nabla f_2, \nabla f_2) g(\nabla f_1, \nabla f_2)^2 - g(\nabla f_2, \nabla f_2)^3) H_\varphi(\nabla f_2, \nabla f_2)] = -\Lambda\varphi\end{aligned}$$

where $\Lambda = 2(\rho - \mathrm{div}N)$, $Q(\nabla f_1, \nabla f_2) = g(\nabla f_1, \nabla f_1) g(\nabla f_2, \nabla f_2) - g(\nabla f_1, \nabla f_2)^2$ and N is the mean curvature vector field of the fibers of f, then (M, g) is locally a warped product $(M_1 \times S^2, g_1 \oplus \psi^2 \mathrm{d}\sigma^2)$, where (M_1, g_1) is a Lorentzian surface and $(S^2, \mathrm{d}\sigma^2)$ is the standard Euclidean 2-sphere. Moreover, (M_1, g_1) can be taken to be of positive curvature in the above warped product decomposition around a point $p \in M$ if $(2\rho + g(N, N))(p) \le 0$.

Proof. Considering the proof of this theorem to be the continuation of the proof of Theorem 1, first we show that each fiber M_2 of f with the induced Riemannian structure from (M, g) is homothetic to the standard Euclidean 2-sphere. For this, we employ the function $\varphi: M \to \mathbb{R}$ given in the statement of the theorem. First note that, if $\overset{M_2}{\nabla}$ denotes both the gradient and

Levi-Civita connection of the induced Riemannian structure of a fiber M_2 of f from (M, g), then, for $X \in \Gamma TM_2$, $g(\nabla^{M_2}\varphi|_{M_2}, X) = X(\varphi|_{M_2}) = X\varphi = g(\nabla\varphi, X)$.

Thus, for $Z, W \in \Gamma(\ker f_*)^{\perp}_{|M_2}$ and $X, Y \in \Gamma TM_2$, such that $\{Z, W, X, Y\}$ is a local orthonormal frame in TM along M_2

$$\begin{aligned}\Delta_{M_2}\varphi|_{M_2} &= \mathrm{div}_{M_2} \overset{M_2}{\nabla}\varphi|_{M_2} = g(\overset{M_2}{\nabla}_X \overset{M_2}{\nabla}\varphi|_{M_2}, X) + g(\overset{M_2}{\nabla}_Y \overset{M_2}{\nabla}\varphi|_{M_2}, Y)\\ &= Xg(\nabla\varphi, X) - g(\nabla\varphi, \overset{M_2}{\nabla}_X X) + Yg(\nabla\varphi, Y) - g(\nabla\varphi, \overset{M_2}{\nabla}_Y Y)\\ &= g(\nabla_X\nabla\varphi, X) + g(\nabla\varphi, \mathbb{I}(X, X)) + g(\nabla_Y\nabla\varphi, Y) + g(\nabla\varphi, \mathbb{I}(Y, Y))\\ &= \Delta\varphi - g(Z, Z)g(\nabla_Z\nabla\varphi, Z) - g(W, W)g(\nabla_W\nabla\varphi, W) + 2g(\nabla\varphi, N)\end{aligned}$$

where Δ_{M_2} and div_{M_2} are the Laplacian and divergence, respectively, in the induced Riemannian structure of M_2 from (M, g). Now since ∇f_1 and ∇f_2 are linearly independent at each $p \in M$, Z and W can be written as a linear combination of ∇f_1 and ∇f_2 along M_2 as follows:

$$\begin{aligned}Z = \frac{1}{Q(\nabla f_1, \nabla f_2)}&[(g(\nabla f_1, \nabla f_1)g(Z, \nabla f_2) - g(\nabla f_1, \nabla f_2)g(Z, \nabla f_1))\nabla f_1\\ &+ (g(\nabla f_2, \nabla f_2)g(Z, \nabla f_1) - g(\nabla f_1, \nabla f_2)g(Z, \nabla f_2))\nabla f_2]\end{aligned}$$

$$\begin{aligned}W = \frac{1}{Q(\nabla f_1, \nabla f_2)}&[(g(\nabla f_1, \nabla f_1)g(W, \nabla f_2) - g(\nabla f_1, \nabla f_2)g(W, \nabla f_1))\nabla f_1\\ &+ (g(\nabla f_2, \nabla f_2)g(W, \nabla f_1) - g(\nabla f_1, \nabla f_2)g(W, \nabla f_2))\nabla f_2]\end{aligned}$$

Then by substituting we obtain the eigenvalue equation

$$\Delta_{M_2}\varphi|_{M_2} = -(\mathrm{Sc})_{M_2}\varphi|_{M_2}$$

on M_2, where $(\mathrm{Sc})_{M_2} = \Lambda_{M_2} = c_{M_2}$ (constant) and $(\mathrm{Sc})_{M_2}$ is the scalar curvature of M_2 in the induced Riemannian structure from (M, g) (see Remark 2).

Also since the eigenvalues of Δ_{M_2} are well known to be negative for the corresponding non-constant eigenfunctions, it follows that $c_{M_2} > 0$. Then a result of Obata ([10], Theorem 5) predicts that M_2 with the induced Riemannian structure from (M, g) is homothetic to the standard Euclidean 2-sphere $(S^2, \mathrm{d}\sigma^2)$ with homothety factor $2/\Lambda_{M_2}$. Thus, by multiplying the warping function ψ^2 of the warped product decomposition $(M_1 \times M_2, g_1 \oplus \psi^2 g_2)$ with the homothety factor $2/\Lambda_{M_2}$ for each fiber of f in this decomposition at corresponding $p_1 \in M_1$, we obtain a local warped product decomposition of (M, g) as $(M_1 \times S^2, g_1 \oplus \tilde{\psi}^2\, \mathrm{d}\sigma^2)$, where $\tilde{\psi}^2 = 2\psi^2/\Lambda$ and $(S^2, \mathrm{d}\sigma^2)$ is the standard Euclidean 2-sphere.

Finally we show that (M_1, g_1) can be taken to be of positive curvature if $(2\rho + g(N, N))(p) \leq 0$ at some $p \in M$. Now let M_1 be an integral manifold of $(\ker f_*)^{\perp}$ passing through p and Ric_{M_1} be the Ricci tensor of M_1 in the induced Lorentzian structure from (M, g). Then from [11] (p. 211), for $Z, W \in \Gamma TM_1$, $\mathrm{Ric}(Z, W) = \mathrm{Ric}_{M_1}(Z, W) - (2/\psi)H_{\psi}(Z, W)$, where ψ^2 is the warping function and H_{ψ} is the Hessian form of ψ on M_1 in its induced Lorentzian structure from (M, g). Now note that, since M_1 is a surface, $\mathrm{Ric}_{M_1}(Z, W) = K_1 g(Z, W)$, where K_1 is its curvature function in its induced Lorentzian structure from (M, g). Also, since $\mathrm{Ric}(Z, W) = T(Z, W) = -\rho g(Z, W)$, it follows that H_ψ is scalar on M_1, that is, $H_\psi(Z, W) = (\Delta\psi/2)g(Z, W)$. Thus $-\rho = K_1 - (\Delta\psi/\psi)$.

Also from [11] (p. 214), since (Sc) $= tr\mathrm{Ric} = trT = 0$

$$0 = 2K_1 + \frac{\Lambda}{\psi^2} - \frac{4\Delta\psi}{\psi} - 2g(N, N)$$

and hence, $K_1 = (\Lambda/2\psi^2) - (2\rho + g(N, N))$. Thus, since $\Lambda > 0$, it follows by the assumption that $K_1 > 0$ on an open neighborhood of $p_1 \in M_1$ in M_1, where $p = (p_1, p_2)$. Then by taking this neighborhood as M_1 in the local warped product decomposition, the claim follows. □

Remark 4. It is important to note that Schwarzschild coordinates $f = (t, r)$ of the Schwarzschild solution do not provide an example to Theorem 2 since they do not satisfy the local Möbius equation. Yet the Kruskal coordinates $f = (u, v)$ of this solution is an example to this theorem since they satisfy the local Möbius equation with $\tau(f) \neq 0$ (that is, f is not harmonic). Yet note that $f = (f_1, f_2)$ taken in the statement of the above theorem need not be completed to a chart on M, even locally.

3.2. Existence of static reference frames

Causal characters of the mean curvature vector field N of the fibers of f also have importance from viewpoint of singularity theorems (see [7], p. 226). In Theorem 2, let B be the open submanifold of (M, g) defined by

$$B = \{p \in M; g(N, N)(p) < 0\}$$

Now we show that, if $B \neq \varnothing$ then it corresponds to the black/white hole region of a static star. For, let M_2 be a fiber of f in B and let $U, V \in \Gamma(TM_2)^\perp$ be null vector fields along M_2 with $g(U, V) = -1$. Also, let L_U and L_V be the shape operators of M_2 with respect to U and V, respectively. Then since $trL_U = 2g(U, N)$ and $trL_V = 2g(V, N)$, it follows that $(trL_U)(trL_V) = -4g(U, N)g(V, N) = -2g(N, N) > 0$, that is, M_2 is a closed trapped surface in B. Hence every fiber of f in B is a closed trapped surface homothetic to the standard Euclidean 2-sphere. Further note that, if (M, g) is vacuum, that is, $\rho = 0$ on M, the Lorentzian factor (M_1, g_1) of the local warped product decomposition of B is of positive curvature.

Also let H_+ and H_- be the subsets of (M, g) defined by

$$H_+ = \{p \in M; N(p) \neq 0 \quad \text{and} \quad g(N, N)(p) = 0\}$$

and

$$H_- = \{p \in M; N(p) \neq 0\}$$

Now, if $H_+ \neq \varnothing$ then it corresponds to the event horizon of the black/white hole region B. Indeed, if M_2 is a fiber of f in H_+ and $U, V \in \Gamma(TM_2)^\perp$ are null vector fields along M_2 with $g(U, V) = -1$, then one of them, say U is a scalar multiple of N at each $p_2 \in M_2$. Thus

$trL_U = 0$ and $trL_V \neq 0$ on M_2. That is, M_2 is a marginally trapped surface in H_+. Hence every fiber of f in H_+ is a marginally trapped surface homothetic to the standard Euclidean 2-sphere. Finally, if $H_- \neq \emptyset$ then it corresponds to a Cauchy horizon as in Reissner solution when $r = r_-$ there. Indeed, if M_2 is a fiber of f in H_- then M_2 is totally geodesic and hence $trL_U = 0 = trL_V$, where U, V are null vector fields orthogonal to M_2 defined as in the above. Again the fibers of f in H_- are homothetic to the standard Euclidean 2-sphere.

Next we analyze the region E of (M, g) which is excluded in the above. That is the open subset of (M, g) defined by

$$E = \{p \in M; g(N, N)(p) > 0\}$$

The fibers of f in E are not trapped surfaces. Indeed, if M_2 is a fiber of f in E, then for null vector fields $U, V \in \Gamma(TM_2)^\perp$ chosen as in the above, we have $(trL_U)(trL_V) = -2g(N, N) < 0$. Consequently, we should expect the existence of a static reference frame ([15], p. 219) in E to consider (M, g) as a space-time describing a static star. To obtain a static reference frame in E, we impose a further condition on N (in fact, on f).

Let (M, g) be a semi-Riemannian manifold and X be a vector field on M. The 'affinity tensor' $\mathcal{L}_X\nabla$ of X is defined by

$$(\mathcal{L}_X\nabla)(U, V) = \mathcal{L}_X\nabla_U V - \nabla_{\mathcal{L}_X U} V - \nabla_U \mathcal{L}_X V$$

for every $U, V \in \Gamma TM$, where $\mathcal{L}$ is the Lie derivative on M (see [13], p. 109). Also, the 'tension field' $\tau(X)$ of X is defined to be the trace of $\mathcal{L}_X\nabla$ with respect to g. A vector field X on a semi-Riemannian manifold (M, g) is called 'affine' if $\mathcal{L}_X\nabla = 0$, and is called 'harmonic-Killing' if $\tau(X) = 0$ (see [2]). Each of these conditions corresponds to the local 1-parameter group of X consists of affine or harmonic maps on (M, g), respectively.

Note that, if the region E in (M, g) is not empty, then (E, g) is a space-time as well.

Lemma 1. *In Theorem 2, suppose $E \neq \emptyset$. Then, $g(\tau(N), Z) = 0$ for $Z \in \Gamma(\ker f_*)^\perp_{|E}$ with $g(N, Z) = 0$ if and only if $g((\mathcal{L}_N\nabla)(Z, Z), Z) = 0$.*

Proof. First note that, since N is pregeodesic (see the last part of Theorem 2, that is, H_ψ is scalar), $(\mathcal{L}_N\nabla)(N, N) = \mathcal{L}_N\nabla_N N \sim N$. Also let $X \in \Gamma(\ker f_*)_{|E}$ with $\mathcal{L}_N X = 0$ (note that, since N is pregeodesic, $(\ker f_*)^\perp_{|E}$ is a totally geodesic distribution and N is the normal curvature vector field of the fibers of f, we can always choose such an $X \in \Gamma(\ker f_*)_{|E}$). Then $(\mathcal{L}_N\nabla)(X, X) = \mathcal{L}_N\nabla_X X = (Ng(X, X))N + V$, where $V \in \Gamma(\ker f_*)_{|E}$.

Now let $N_1 = N/(g(N, N))^{1/2}$ and let $Z_1, N_1 \in \Gamma(\ker f_*)^\perp_{|E}$ and $X, Y \in \Gamma(\ker f_*)_{|E}$ be such that $\{Z, N_1, X, Y\}$ is a local orthonormal frame in TM. Then by the tensoriality of $\mathcal{L}_N\nabla$

$$\begin{aligned} g(\tau(N), Z) &= g(Z_1, Z_1)g((\mathcal{L}_N\nabla)(Z_1, Z_1), Z) + g(N_1, N_1)g((\mathcal{L}_N\nabla)(N_1, N_1), Z) \\ &\quad + g(X, X)g((\mathcal{L}_N\nabla)(X, X), Z) + g(Y, Y)g((\mathcal{L}_N\nabla)(Y, Y), Z) \\ &= g(Z_1, Z_1)g((\mathcal{L}_N\nabla)(Z_1, Z_1), Z) \end{aligned}$$

Thus the claim follows. □

Lemma 2. *In* Theorem 2, *the normal curvature vector field N has the following properties on* $B \cup H_+ \cup E$.

1. *The* 1-*form* $\omega(\cdot) = g(\cdot, N)$ *is exact*.
2. $Zg(N, N) = 0$ *and* $[N, Z] = \eta Z$ *for every* $Z \in \Gamma(\ker f_*)^{\perp}$ *with* $g(Z, Z) = c$ (*constant*) *and orthogonal to N*, *where* η *is a function on* $B \cup H_+ \cup E$ *which is constant on each fiber of f*.

Proof.

1. Note that $\omega(X) = 0$ for every $X \in \Gamma \ker f_*$ and $\omega(Z) = g(Z, N) = g(Z, -(\nabla\psi/\psi)) = -(1/\psi)g(Z, \nabla\psi)$ for $Z \in \Gamma(\ker f_*)^{\perp}$. Thus $\omega = d\log(1/\psi)$.
2. Since ω is exact by (1)

$$\begin{aligned}0 &= 2d\omega(N, Z) = N\omega(Z) - Z\omega(N) - \omega([N, Z])\\ &= -2g(\nabla_Z N, N) - g([N, Z], N)\\ &= -2g(\nabla_N Z, N) - 2g([Z, N], N) - g([N, Z], N)\\ &= g(\nabla_Z N, N) - g(\nabla_N Z, N) = \tfrac{1}{2}Zg(N, N)\end{aligned}$$

Hence it also follows that $[N, Z] = \eta Z$, where η is a function on *M*. Note that, since *Z* is the lift of its restriction to an integral manifold of $(\ker f_*)^{\perp}$, as well as *N*, η is constant on each fiber of *f* (this can also be shown by using the form of the curvature tensor ([11], p. 210)). □

Theorem 3. *In* Theorem 2, *suppose* $E \neq \emptyset$. *If* $g(\tau(N), Z) = 0$ *for every* $Z \in \Gamma(\ker f_*)^{\perp}_{|E}$ *orthogonal to N then there exists a unique static reference frame* $Z_1 \in \Gamma(\ker f_*)^{\perp}_{|E}$ *orthogonal to N*. *Conversely, if there is a stationary reference frame* $Q_1 \in \Gamma(\ker f_*)^{\perp}_{|E}$ *orthogonal to N then* $g(\tau(N), Z) = 0$ *for every* $Z \in \Gamma(\ker f_*)^{\perp}_{|E}$ *orthogonal to N and* Q_1 *is a static reference frame*.

Proof. Let $Z_1 \in \Gamma(\ker f_*)^{\perp}_{|E}$ be a unit time-like vector field orthogonal to *N*. Then by Lemma 2, $[N, Z_1] = \eta Z_1$ for some function η on *E* which is constant on each fiber of *f*. Now we show that there exists a function λ defined on each sufficiently small open set in *E* such that $[N, \lambda Z_1] = 0$ and $\mathrm{d}\lambda(Z_1) = 0$. For, consider the equation

$$0 = [N, \lambda Z_1] = (N\lambda)Z_1 + \lambda[N, Z_1] = [(N\lambda) + \lambda\eta]Z_1$$

Hence it suffices to show that $N\lambda + \lambda\eta = 0$ locally has a solution λ with $Z_1\lambda = 0$. First note that, $\nabla_{Z_1} Z_1 \in \Gamma(\ker f_*)^{\perp}_{|E}$ is orthogonal to Z_1 and, $[N, Z_1]$ is orthogonal to *N* by Lemma 2. Also by Lemma 1, $g((\mathcal{L}_N \nabla)(Z_1, Z_1), Z_1) = 0$, and we have

$$\begin{aligned}-Z_1\eta &= Z_1 g(\mathcal{L}_N Z_1, Z_1) = g(\nabla_{Z_1}\mathcal{L}_N Z_1, Z_1)\\ &= g(\mathcal{L}_N \nabla_{Z_1} Z_1, Z_1) - g(\nabla_{\mathcal{L}_N Z_1} Z_1, Z_1) = 0\end{aligned}$$

Now for the solution λ, let γ be an integral curve of Z_1 and let φ_t be the local 1-parameter group of *N*. Note that, since *N* is pregeodesic, $\varphi_t \circ \gamma$ is also an integral curve of Z_1 (up to a parametrization). Hence the function λ defined locally by $\lambda(p) = \mathrm{e}^{-\int_0^t \eta\circ\varphi_s(q)\mathrm{d}s}$, where q is on γ with $\varphi_t(q) = p$, then satisfies $N\lambda + \lambda\eta = 0$ with $Z_1\lambda = 0$ and which is constant on each fiber of f. In fact, $Z = \lambda Z_1$ is a Killing vector field. For, it suffices to show that ∇Z is skew-adjoint in $(\ker f_*)^\perp$ since $\nabla_X Z = 0$ for every $X \in \Gamma \ker f_*$. Indeed

$$g(\nabla_N Z, Z) = g(\nabla_Z N, Z) = -g(N, \nabla_Z Z)$$

$$g(\nabla_N Z, N) = -g(Z, \nabla_N N) = 0$$

$$g(\nabla_Z Z, N) = -g(Z, \nabla_Z N) = -g(Z, \nabla_N Z)$$

$$g(\nabla_Z Z, Z) = \tfrac{1}{2} Zg(Z, Z) = 0$$

that is, Z is a Killing vector field. Without loss of generality, we may assume that $Z_1 = (1/\lambda)Z$ is future-directed and hence, is a stationary reference frame. In fact, Z_1 is a static reference frame, since $[N, Z] = 0$, there exists a (local) chart (t, r, θ, ϕ) for (E, g), where (θ, ϕ) is a chart for S^2, such that $Z = \partial/\partial t$ and $N = \partial/\partial r$. Thus it follows that $Z = -\lambda^2 \nabla t$ and $Z_1 = -\lambda \nabla t$. Note that the static reference frame Z_1 is defined locally. To show that it can be defined on E, it suffices to show that it is locally unique. Let $Q \in \Gamma(\ker f_*)^\perp_{|E}$ Killing vector field orthogonal to N where Z_1 is defined. Hence there exists a function h such that $Z = hQ$. In fact, this function is constant, since

$$\begin{aligned} Ng(Z, Z) &= (Nh^2)g(Q, Q) + h^2 Ng(Q, Q) = (Nh^2)g(Q, Q) + 2hg(\nabla_N Q, Z) \\ &= (Nh^2)g(Q, Q) - 2hg(N, \nabla_Z Q) = (Nh^2)g(Q, Q) - 2g(N, \nabla_Z Z) \\ &= (Nh^2)g(Q, Q) + Ng(Z, Z) \end{aligned}$$

$Nh = 0$, and since

$$0 = Zg(Z, Z) = (Zh^2)g(Q, Q) + Zg(Q, Q) = (Zh^2)g(Q, Q)$$

$Zh = 0$, and it follows that h is constant, because $\nabla_X Z = 0$ for every $X \in \Gamma \ker f_*$. Thus Z_1 is locally unique in $(\ker f_*)^\perp_{|E}$.

Now, note that the domains of the locally defined Z_1 can be chosen as open “rectangles” which are products of some integral curves of N, Z_1 and fibers of f. These domains have the property that, if $\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3$ are such rectangles with non-empty $\mathfrak{R}_1 \cap \mathfrak{R}_2$, $\mathfrak{R}_1 \cap \mathfrak{R}_3$ and $\mathfrak{R}_2 \cap \mathfrak{R}_3$, then $\mathfrak{R}_1 \cap \mathfrak{R}_2 \cap \mathfrak{R}_3$ is also a non-empty such rectangle. Thus, it is possible to define a function $\tilde{\lambda}$ on E such that $\tilde{Z} = \tilde{\lambda} Z_1$ is a Killing vector field on E. This can be done by starting with a Killing vector field on a rectangle and extending it nearby intersecting rectangles by multiplying the Killing vector fields on nearby rectangles with an appropriate constant. Note here that the resulting static reference frame on E need not be synchronizable (see [15], p. 53).

Conversely, let $Q_1 \in \Gamma(\ker f_*)^\perp_{|E}$ be a stationary reference frame orthogonal to N and Q be a corresponding Killing vector field on E. First note that Q_1 is then necessarily a static reference frame since $\nabla_X Q_1 = 0$ for every $X \in \Gamma(\ker f_*)_{|E}$. To show that $g(\tau(N), Q) = 0$, it suffices to show that $[N, Q] = 0$. Indeed it follows from

$$g((\mathcal{L}_N \nabla)(Q, Q), Q) = g(\mathcal{L}_N \nabla_Q Q, Q) - g(\nabla_Q \mathcal{L}_N Q, Q) - g(\nabla_{\mathcal{L}_N Q} Q, Q) = 0$$

since $\nabla_Q Q$ is orthogonal to Q, that $g(\tau(N), Q) = 0$ by Lemma 1. To show $[N, Q] = 0$, note that

$$g([N, Q], Q) = g(\nabla_N Q, Q) - g(\nabla_Q N, Q) = -g(N, \nabla_Q Q) + (N, \nabla_Q Q) = 0$$

and

$$g([N, Q], N) = g(\nabla_N Q, N) + g(\nabla_Q N, N) = 0$$

Hence it follows that $[N, Q] = 0$. □

Remark 5. The (local) chart (t, r, θ, ϕ) constructed in the proof of Theorem 3 can be considered as a "Schwarzschild type" coordinate system on E. Note also that, by Lemma 2, the warping function ψ^2 only depends on r in this coordinate system since $Zg(N, N) = 0$, where $Z = \partial/\partial t$ and $N = \partial/\partial r$. Also the induced metric tensor on an integral manifold M_1 of $(\ker f_*)^\perp$ can locally be written as

$$g_1 = g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) \mathrm{d}t \otimes \mathrm{d}t \oplus g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) \mathrm{d}r \otimes \mathrm{d}r$$

Furthermore, since

$$\frac{\partial}{\partial t} g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = 0 = \frac{\partial}{\partial t} g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right),$$

$$g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)$$

and

$$g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right)$$

only depend on r in this coordinate system. Also the coordinate r is related with the Euclidean radius of the fibers of f. Recall from the proof of Theorem 2 that the Euclidean radius of a fiber of f is given by $\sqrt{2/\Lambda}$ and $\Lambda = 2(\rho - \mathrm{div} N)$ (where $N = \partial/\partial r$). Now it is easy to see that $\mathrm{div} N$ only depends on r in this coordinate system. Also from $0 = (\mathrm{div} T)(Z) = \mathrm{d}\rho(Z)$, the negative Faraday stresses only depend on r as well (note here that $(\mathrm{div} T)(N) = \mathrm{d}\rho(N) + 2\rho g(N, N)$, and hence, $\mathrm{d}\rho(N) = 0$ if and only if $\rho = 0$, point-wise on E). Consequently, $\sqrt{2/\Lambda}$ only depends on r in this coordinate system, and hence, r is locally a function of $\sqrt{2/\Lambda}$. As an example, in the Schwarzschild coordinate system (t, r, θ, ϕ) for the region E of the Reissner solution, the normal curvature vector field N is given by

$$N = -\frac{\nabla r}{r} = -\frac{1}{r}\left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right) \frac{\partial}{\partial r}$$

In fact, the coordinate r in this coordinate system is nothing but locally scaling the corresponding coordinate in the "Schwarzschild type" coordinate system constructed above, to make the warping function as simple as $\psi^2(r) = r^2$ in this coordinate system. Finally note that, the normal curvature vector field N in the Reissner solution is not harmonic-Killing, that is, $\tau(N) \neq 0$, yet $g(\tau(N), Z) = 0$, where $Z = \partial/\partial t$.

Remark 6. In [6], a vector field N on a semi-Riemannian manifold (M, g) is called 'weakly affine' if $g((\mathcal{L}_N \nabla)(U, V), V) = 0$ for every U, V orthogonal to N. In the case N is the mean curvature vector field in Theorem 2, N is weakly affine on the region E if and only if $g((\mathcal{L}_N \nabla)(Z, Z), Z) = 0$ for every $Z \in \Gamma(\ker f_*)^{\perp}_{|E}$ orthogonal to N (see [6]). Hence by Lemma 1, N is weakly affine if and only if $g(\tau(N), Z) = 0$ for every $Z \in \Gamma(\ker f_*)^{\perp}_{|E}$ orthogonal to N.

Acknowledgements

The third-named author (D.N.K.) is grateful to the Abdus Salam ICTP for its support in the preparation of the early version of this paper. Also we wish to thank Mattias Blau for calling our attention to Obata's equation. The first and second author acknowledge support by projects DGESIC PB97-0504-C02-01 and PGIDT99 PX120703B (Spain).

References

[1] J.K. Beem, P.E. Ehrlich, K.L. Easley, Global Lorentzian Geometry, 2nd Edition, Marcel Dekker, New York, 1996.
[2] C.T.J. Dodson, T. Pérez, M.E. Vázquez-Abal, Harmonic-Killing vector fields, Bull Belgian Math. Soc., to appear.
[3] M. Fernández-López, E. Garcia-Rio, D.N. Kupeli, A characterization of cosmological time functions, Ann. Global Anal. Geom. 20 (2001) 1–13.
[4] T. Frankel, Gravitational Curvature, Freeman, San Francisco, 1979.
[5] E. García-Río, D.N. Kupeli, Semi-Riemannian Maps and Their Applications, Kluwer Academic Publishers, Dordrecht, 1999, Math. Appl. 475.
[6] E. García-Río, D.N. Kupeli, A local characterization of Schwarzschild and Reissner metrics, Gen. Rel. Grav. 26 (1994) 1233–1248.
[7] S.W. Hawking, G.F.R. Ellis, The Large Scale Structure of Space-Time, Cambridge University Press, Cambridge, 1973.
[8] W. Kühnel, H.-B. Rademacher, Conformal vector fields on pseudo-Riemannian spaces, Differential Geo. Appl. 7 (1997) 237–250.
[9] T. Nore, Second fundamental form of a map, Ann. Mat. Pura Appl. 146 (4) (1987) 281–310.
[10] M. Obata, Certain condition for a Riemannian manifold to the isometric with a sphere, J. Math. Soc. Jpn. 14 (1962) 333–340.
[11] B. O'Neill, Semi-Riemannian Geometry With Applications to Relativity, Academic Press, New York, 1983.
[12] B. Osgood, D. Stowe, The Schwarzian derivative and conformal mapping of Riemannian manifolds, Duke Math. J. 67 (1992) 57–99.
[13] W.A. Poor, Differential Geometric Structures, McGraw-Hill, New York, 1981.
[14] R. Ponge, H. Reckziegel, Twisted products in pseudo-Riemannian geometry, Geo. Dedicata 48 (1993) 15–25.
[15] R.K. Sachs, H. Wu, General Relativity for Mathematiciens, Springer, New York, 1977.